Spin Geometry, by Lawson & Michelsohn

Note by Conan Leung

(II) Spin Geometry & Dirac operators.

(X^n, g)

$\leftrightarrow$ $\mathbb{R}^n \longrightarrow TX \longrightarrow X$ w/ fiberwise inner product.

$\rightsquigarrow$ $O(n) \longrightarrow P_{O(n)}(X) \longrightarrow X$ frame bundle

is a principal $O(n)$-bundle.

Reason. Given (V, g) inner product space.

frame (= orthonormal base $e_1, \dots, e_n$

$\Longleftrightarrow$ $(V, g) \xrightarrow[\text{isometry}]{\sim} (\mathbb{R}^n, g_{std})$

$e_1 \longleftrightarrow (1, 0, 0, \dots)$ etc.

Linear alg: 1) $\{\text{frames on } (V, g)\} \circlearrowleft O(n)$ simply transitive.

2) $V \simeq \{\text{ — '' — }\} \times_{O(n)} \mathbb{R}^n$

Can recover (TX, g) as associated bundle via repr. $O(n) \curvearrowright \mathbb{R}^n$, i.e.

$$TX = P_{O(n)}(X) \times_{O(n)} \mathbb{R}^n$$

- If M is oriented $\rightsquigarrow$ $P_{SO(n)}(X) = \left\{\begin{matrix}\text{oriented} \\ \text{frames}\end{matrix}\right\}$

  $\rightsquigarrow$ principal $SO(n)$-bundle.

$SO(n) \hookrightarrow O(n) \Rightarrow P_{SO(n)}(X)$ determines $P_{O(n)}(X)$.

- In general, given pr. G-bdl : $G \to P \to X$

$G \to H \rightsquigarrow$ pr. H-bdl : $H \to P \times_G H \to X$

Equivalently,

$$\begin{tikzcd} & G \arrow[r] \arrow[d] & H \arrow[d] & \\ & E_G \arrow[r, equal] \arrow[d] & E_H & \sim * \\ X \arrow[r, "f_P", "\wr\ G\text{-bdl.}"'] & B_G \arrow[r] & B_H \arrow[from=u, crossing over] & \end{tikzcd}$$

- The converse has topo. constraints.
- If $w_1(X) = 0 \in H^1(X, \mathbb{Z}_2)$, then

$\exists P_{SO(n)}(X) \rightsquigarrow P_{O(n)}(X)$ (i.e. oriented)

- If $w_2(X) = 0 \in H^2(X, \mathbb{Z}_2)$, then

$\exists P_{Spin(n)}(X) \rightsquigarrow P_{SO(n)}(X)$ (i.e. Spin)

Fact (1) $X^3 \Rightarrow w_2 = w_1^2$ (So, oriented $\Rightarrow$ spin)

(2) X complex. $w_1(X) \equiv c_1(X) \pmod 2$

Eg. $c_1(\mathbb{CP}^n) = n+1 \in \mathbb{Z} \simeq H^2(\mathbb{CP}^n, \mathbb{Z})$

$\mathbb{CP}^n$ Spin $\iff n \in 2\mathbb{Z}+1$

Eg. $X^n = \{f = 0\} \subseteq \mathbb{CP}^{n+1}$ $\quad n \geq 2$

$c_1(X) = n + 2 - \deg f$

X Spin $\iff n + \deg f \in 2\mathbb{Z}$.

Eg. X^4, $\pi_1 = 0$, cpt.

X Spin $\Longrightarrow \forall c \in H^2(X, \mathbb{Z}), c \cdot c \in 2\mathbb{Z}$

$\xRightarrow{+C^\infty} b_2^+ - b_2^- \in 16\mathbb{Z}$

(w/o $C^\infty \Rightarrow b_2^+ - b_2^- \in 8\mathbb{Z}$)

(X^n, g, ν) oriented Riem. mfd.

$\rightsquigarrow SO(n) \to P_{SO(n)}(X) \to X$ ori. frame bdl.

- $SO(n) \curvearrowright \Lambda^k \mathbb{R}^{n*} \rightsquigarrow \Lambda^k T_X^*$
- $SO(n) \curvearrowright Cl(\mathbb{R}^n) \rightsquigarrow Cl(X)$ Clifford bdl.

$$Cl(X) \underset{\text{isometry}}{\cong} \Lambda^\bullet T_X^* \quad \text{as vector bundles.}$$

- If X^n Spin $(\rightsquigarrow P_{Spin(n)}(X))$

$$Spin(n) \hookrightarrow Cl^0(\mathbb{R}^n) \curvearrowright M \rightsquigarrow M \to \$(X) \to X$$

real spinor bdl.

- $Cl(X) \curvearrowright \$(X)$ bdl. of modules over bdl. of alg.

$/\mathbb{C}$. $Spin(n) \hookrightarrow Cl^0(\mathbb{R}^n) \xhookrightarrow{\otimes\mathbb{C}} Cl^0(\mathbb{C}^n) \curvearrowright M_{\mathbb{C}} \rightsquigarrow \$_{\mathbb{C}}(X)$

n	1	2	3	4	5	6	7	8	9
Cl_n^0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$2\mathbb{R}(8)$	$\mathbb{R}(16)$
irred. $\$/\$^\pm$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}_\pm$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8_\pm$	$\mathbb{R}^{16}$
	$\otimes\mathbb{C}$		$\Vert$ as v.s.	$\Vert$	$\Vert$		$\otimes\mathbb{C}$	$\otimes\mathbb{C}$	$\otimes\mathbb{C}$
$/\mathbb{C}$ $\$_{\mathbb{C}}/\$_{\mathbb{C}}^\pm$	$\mathbb{C}$	$\mathbb{C}_\pm$	$\mathbb{C}^2$	$\mathbb{C}^2_\pm$	$\mathbb{C}^4$	$\mathbb{C}^4_\pm$	$\mathbb{C}^8$	$\mathbb{C}^8_\pm$	$\mathbb{C}^{16}$

Connection.

$$\begin{array}{lllll} \omega \text{ on} & G \to P \to X & & \text{pr. } G\text{-bdl.} \\ \rightsquigarrow & \quad\ \rightsquigarrow\ G \curvearrowright \mathbb{R}^n & & \\ \nabla \text{ on} & \mathbb{R}^n \to E \to X & & \text{assoc. VB} \end{array}$$

$$R := \nabla^2 \in \Omega^2(X, \operatorname{End} E) \qquad \text{curvature.}$$

i.e. $R_{v,w} = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]}$

(E, g) $\quad \nabla$ compatible w/ $g \in \Gamma(\operatorname{Sym}^2 E)$

$\iff \nabla g = 0 \quad \in \Omega^1(X, \operatorname{Sym}^2 E)$

$\iff d\langle e, e' \rangle = \langle \nabla e, e' \rangle + \langle e, \nabla e' \rangle$

$\Rightarrow R \in \Omega^2(X, \operatorname{ad} E) \qquad \underset{\underset{\textstyle\Lambda^2 E}{\cong}}{\mathfrak{o}(E)} \to \operatorname{ad} E \to X$

(i.e. $\langle R_{v,w} e, e' \rangle + \langle e, R_{v,w} e' \rangle = 0$)

Assume E Spin $\rightsquigarrow$ $\quad Cl(E) \curvearrowright \$(E)$

$$\begin{array}{ccc} Cl(E) & & \$(E) \\ \downarrow{\scriptstyle Cl(\mathbb{R}^n)} & & \downarrow{\scriptstyle M} \\ X & = & X \end{array}$$

$R \in \Omega^2(X, \operatorname{Der}(Cl(E)))$ $\quad$ i.e. $R(\varphi \cdot \psi) = (R\varphi)\cdot\psi + \varphi \cdot (R\psi)$

$\qquad\qquad \cap$ $\qquad\qquad\qquad\qquad\qquad\qquad$ Clifford multi.

$\qquad \operatorname{End}(Cl(E))$

(Any ∇ & R respect Clifford multi. ($\because \nabla g = 0$)).

- On $\$(E)$, $\quad R(\sigma) = \frac{1}{2} \sum_{i<j} R_{ij}\, e_i e_j \cdot \sigma = \frac{1}{4} \sum_{i,j} R_{ij}\, e_i e_j \cdot \sigma$

Pf: $\Lambda^* T^* \xrightarrow{\sim} Cl_n$ $\Big\} \rightsquigarrow \underbrace{\mathrm{spin}(n)}_{\mathrm{so}(n)} \curvearrowright Cl_n$

$\qquad x \wedge y \longleftrightarrow \frac{1}{4}[x, y]$ $\qquad x \wedge y \rightsquigarrow Ad_*(x \wedge y) = ad_{\frac{1}{4}[x,y]}$

Conn. $\omega = -\sum_{i<j} \underbrace{\omega_{ij}}_{\text{conn. 1-form}} e_i \wedge e_j \rightsquigarrow -\sum_{i<j} \omega_{ij} \underbrace{\tfrac{1}{4}[e_i, e_j]}_{\frac{1}{2} e_i \cdot e_j \cdot} \cdot$

$\Rightarrow \nabla^{\$} \sigma = \frac{1}{2} \sum_{i<j} \omega_{ij} \otimes e_i \cdot e_j \cdot \sigma$

$\Rightarrow R^{\$} \sigma = \frac{1}{2} \sum_{i<j} \Omega_{ij} \otimes e_i \cdot e_j \cdot \sigma$

Dirac operator.

(X, g) Spin

Take $E = T^*_X \rightsquigarrow \exists!\, \nabla,\ \nabla g = 0 = \mathrm{Tor}(\nabla)$

$$\not{D} : \Gamma(\$(X)) \xrightarrow{\nabla} \Gamma(T^*_X \otimes \$(X)) \xrightarrow{\ \cdot\ } \Gamma(\$(X))$$

$$\langle \not{D}\sigma, \eta \rangle \triangleq \langle e_j \nabla_{e_j} \sigma, \eta \rangle$$

$$= \langle \underbrace{e_j \cdot e_j}_{-1} \nabla_{e_j} \sigma, e_j \cdot \eta \rangle$$

$$\overset{\nabla g = 0}{=} - \underbrace{e_j \underbrace{\langle \sigma, e_j \cdot \eta \rangle}_{\langle -V, e_j \rangle}}_{\langle \nabla_{e_j} V, e_j \rangle + \langle V, \cancel{\nabla_{e_j} e_j} \rangle} + \langle \sigma, \underbrace{\nabla_{e_j}(e_j \cdot \eta)}_{(\cancel{\nabla_{e_j} e_j}) \cdot \eta + e_j \cdot \nabla_{e_j} \eta} \rangle$$

$$\underbrace{\langle \nabla_{e_j} V, e_j \rangle}_{\mathrm{div}\, V}, \qquad (\mathrm{Tor}\nabla = 0), \qquad \underbrace{e_j \cdot \nabla_{e_j} \eta}_{\not{D}\eta}$$

$$\mathrm{div} : \Gamma(T_X) \xrightarrow{\lrcorner\, vol} \Omega^{n-1}(X) \xrightarrow{d} \Omega^n(X) \xleftarrow{\lrcorner\, vol} \Omega^0(X)$$

$$\int (\mathrm{div}\, V)\, vol = \int d(V \lrcorner\, vol) \overset{\text{Stokes'}}{=} 0 \quad (\text{if } \partial X = \phi)$$

$$\Rightarrow \int_X \langle \not{D}\sigma, \eta \rangle = \int_X \langle \sigma, \not{D}\eta \rangle \quad \text{formally adjoint.}$$

If $\not{D}^2 \sigma = 0$

$$\Rightarrow 0 = \int \langle \sigma, \not{D}^2 \sigma \rangle \overset{\text{above}}{=} \int \langle \not{D}\sigma, \not{D}\sigma \rangle = \int |\not{D}\sigma|^2$$

$$\Rightarrow \not{D}\sigma = 0$$

i.e. $\mathrm{Ker}\, \not{D} = \mathrm{Ker}\, \not{D}^2$.

Twisted Dirac operator.

coupled w/ VB $\mathbb{R}^k \to E \to X$ w/ ∇_E

$\rightsquigarrow \not{D}_E : \Gamma(X, \$ \otimes E) \circlearrowleft$

- If $E = \$ \Rightarrow \$ \otimes E = Cl(X) \simeq \Lambda^\bullet T^*_X$

Theorem. $$\begin{array}{ccc} \not{D}_{E=\$} & : & \Gamma(Cl(X)) \circlearrowleft \\ \| & & \| \\ d + d^* & : & \Omega^\bullet(X) \circlearrowleft \end{array}$$

$$\begin{array}{ccccc} Cl(X) & = & Cl^0(X) & \oplus & Cl^1(X) \\ \| & & \| & & \| \\ \Lambda^\bullet T^* & = & \Lambda^{ev} T^* & + & \Lambda^{odd} T^* \end{array}$$

$$\not{D} = \begin{pmatrix} 0 & \not{D}_1 \\ \not{D}_0 & 0 \end{pmatrix}$$

$$\mathrm{Index}\,\not{D} := \dim \underset{\Lambda^{ev} T^*}{\underset{\cap}{\mathrm{Ker}\,\not{D}_0}} - \overbrace{\dim \underset{\Lambda^{od} T^*}{\underset{\cap}{\mathrm{Ker}\,\not{D}_1}}}^{\dim \mathrm{Coker}\,\not{D}_0}$$

$$= \chi(X) \quad \text{by Hodge thm.}$$

- wrt a different splitting $Cl = Cl^+ \oplus Cl^-$

$\rightsquigarrow \mathrm{Index}\,\not{D} = \mathrm{Sign}(X)$.

- Use $\$_{\mathbb{C}} \longrightarrow$ Atiyah-Singer index theorem.

$$\mathrm{Index}\,\not{D}_E = \int_X \hat{A}(X)\, ch(E).$$

Bochner idenity.

Given (E, ∇_E) on spin mfd (X, g).

$$\begin{array}{ccc}
\Gamma(\$ \otimes E) & \underset{\not{D}_E^* = \not{D}_E}{\overset{\not{D}_E}{\rightleftarrows}} & \Gamma(\$ \otimes E) \\
\| & & \\
\Omega^0(\$ \otimes E) & \underset{\nabla_E^*}{\overset{\nabla_E}{\rightleftarrows}} & \Omega^1(\$ \otimes E)
\end{array}$$

$(\because$ self-adj$)$

Theorem. $\not{D}^2 = \nabla^* \nabla + \mathcal{R}$

where $\mathcal{R}(\varphi) = \frac{1}{2} e_j \cdot e_k R_{e_j, e_k}(\varphi)$

Pf: $\not{D}^2 = \sum_{j,k} e_j \cdot \nabla_{e_j}(e_k \cdot \nabla_{e_k})$

$= \sum_{j,k} e_j \cdot e_k \nabla_{e_j} \nabla_{e_k}$ $\quad$ ($\nabla_{e_j} e_k = 0$ in normal coord.)

$= \sum_{j=k} + \sum_{j<k} + \sum_{j>k}$ $\quad$ $(\because e_j \cdot e_k = - e_k \cdot e_j)$

$= \underbrace{- \sum_j \nabla_{e_j} \nabla_{e_j}}_{\nabla^* \nabla} + \underbrace{\sum_{j<k} e_j \cdot e_k (\nabla_{e_j} \nabla_{e_k} - \nabla_{e_k} \nabla_{e_j})}_{\mathcal{R}}$

Say $E = \$$, then $\$_E = C\ell = \Lambda^\bullet T^* \supset T^*$

Take $\varphi \in \Gamma(T^*) = \Omega^1(X)$

$$\mathcal{R}(\varphi) = \frac{1}{2} \sum_{\substack{i,j \\ (\text{or } i \neq j)}} e_i \cdot e_j \cdot R_{e_i, e_j}(\varphi)$$

$$\left(R_{\underbrace{ij}k\ell} + R_{\underbrace{jk}i\ell} + R_{\underbrace{ki}j\ell} = 0 \;\;\&\;\; R_{\underline{ij}\,\underline{k\ell}} = R_{\underline{k\ell}\,\underline{ij}} \right)$$

$$R_{e_i e_j}(\varphi) = \sum_k \langle R_{e_i, e_j}(\varphi), e_k \rangle e_k = \sum_k R_{ij\varphi k} e_k$$

$$= \underbrace{\sum_{k \neq i,j} R_{ij\varphi k} e_k}_{} + R_{ij\varphi i} e_i + R_{ij\varphi j} e_j$$

$$3 \sum_{\substack{i \neq j \\ \neq k \neq}} R_{ij\varphi k}\, e_i \cdot e_j \cdot e_k \xlongequal{\text{rearranging indices}} \sum_{\dots} \underbrace{(R_{ijk\varphi} + R_{kij\varphi} + R_{jki\varphi})}_{= 0 \text{ (Bianchi)}} e_k$$

$$\Rightarrow \mathcal{R}(\varphi) = \frac{1}{2} \sum_{i \neq j} \Big(R_{ij\varphi i} \underbrace{e_i \cdot e_j \cdot e_i}_{-e_j} + R_{ij\varphi j}\, e_i \cdot \underbrace{e_j \cdot e_j}_{-1} \Big)$$

same sum

$$= - \sum_{i \neq j} R_{ij\varphi j}\, e_i = \mathrm{Ric}(\varphi).$$

Theorem. (X, g) closed w/ $\mathrm{Ric} > 0 \Rightarrow b_1 = 0$

Pf: $b_1 \neq 0 \overset{\text{Hodge}}{\Longrightarrow} \exists \varphi \in \Omega^1 \setminus 0$, $\underbrace{\Delta}_{\not{D}^2}\varphi = 0$

$$\overset{\text{Bochner}}{\Longrightarrow} \int \langle \nabla^*\nabla\varphi + \mathrm{Rc}(\varphi), \varphi \rangle = 0$$

$$= \int |\nabla\varphi|^2 + \int \langle \mathrm{Rc}(\varphi), \varphi \rangle$$

$$\Rightarrow \quad \nabla\varphi = 0 \;\;\&\;\; \varphi = 0$$

Similarly, if $0 < Rm \in \Gamma(X, Sym^2(\Lambda^2 T^*))$

(i.e. positive curvature operator)

then $b_k = 0 \quad \forall\, k = 1, \cdots, n-1$

(i.e. $H^\bullet(X, \mathbb{Q}) \cong H^\bullet(S^n, \mathbb{Q})$, rational homology sphere)

When $\nexists E$,

$$\not{D}^2 = \nabla^* \nabla + \frac{1}{4} K$$

scalar curv. $Tr(Rc)$.

In particular, X Spin w/ $K > 0$

$\Rightarrow \quad Ker\, \not{D} = 0$

$\Rightarrow \quad 0 = Index\, \not{D} = \int_X \hat{A}(X)$

Atiyah-Singer index thm.

Pf: Curv. $R^{\$}_{v,w} = \frac{1}{4} \sum_{k,l} \langle R_{v,w}(e_k), e_l \rangle e_k \cdot e_l \cdot \; : \$ \longrightarrow \$$

$$\mathcal{R} = \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot R^{\$}_{e_i, e_j} = \frac{1}{8} \sum_{i,j,k,l} R_{ijkl}\, e_i \cdot e_j \cdot e_k \cdot e_l$$

$$= \frac{1}{8} \sum_l \Big[\frac{1}{3} \sum_{i \neq j \neq k} \underbrace{(R_{ijkl} + R_{kijl} + R_{jkil})}_{0 \text{ (Bianchi)}} e_{ijk} + \sum_{i,j} \big(\underbrace{R_{ijil}\, e_{iji} + R_{ijjl}\, e_{ijj}}_{\text{same}}\big) e_l \Big]$$

$$= \frac{1}{4} \sum_{i,j,l} R_{ijil}\, e_j \cdot e_l = \frac{1}{4} \sum_{i,j} R_{ijij} (-1) = \frac{1}{4} K.$$